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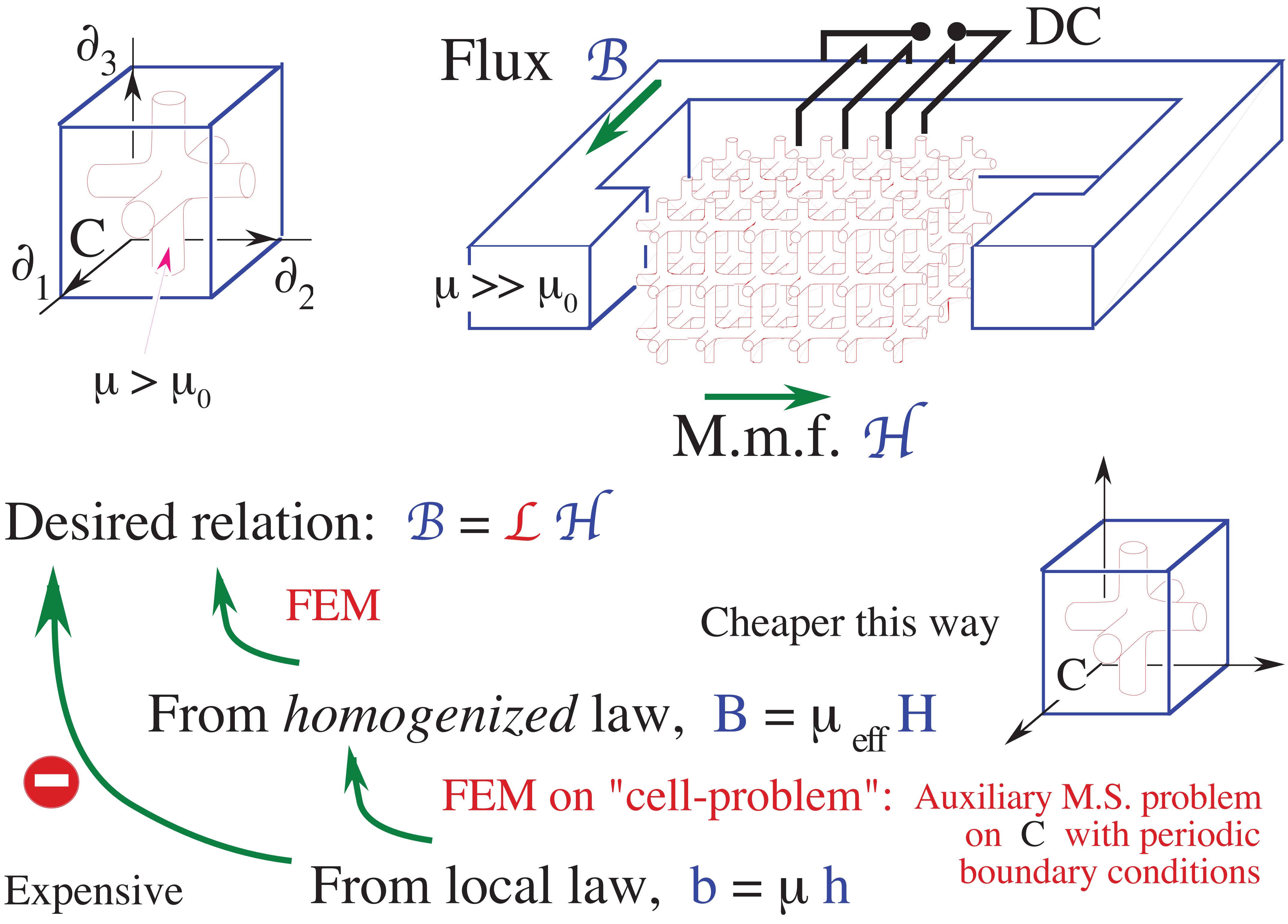
Proving homogenization correct

Statics, easy. Full Maxwell, tricky.

Homogenization

What it is,

what kind of justification it requires

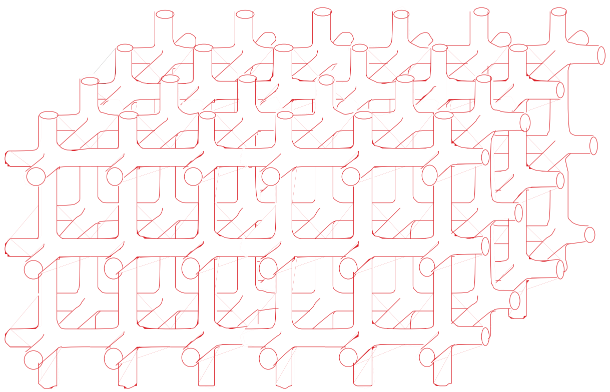


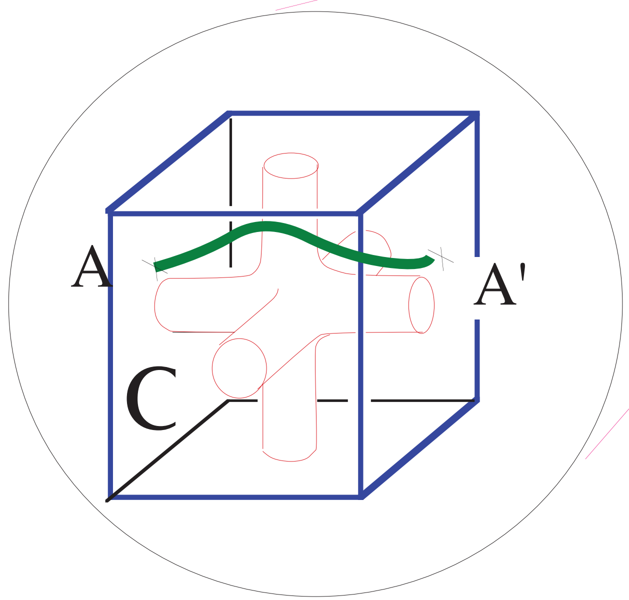
∂3
C
∂1
∂2
μ > μ0
Flux B
DC
μ >> μ0
M.m.f. H
Desired relation: B = L H
FEM
Cheaper this way
C
From homogenized law, B = μ eff H
FEM on "cell-problem": Auxiliary M.S. problem
on C with periodic
boundary conditions
Expensive
From local law, b = μ h

The cell problem

actual field = large-scale average + "C-periodic" correction

h $\quad$ H $\quad$ $\mathrm{grad}\,\varphi$





H considered *constant*
(i.e., a vector)
over the cell

$$\varphi(A) = \varphi(A')$$

("C-periodic")

$$\int \mu |H + \nabla\varphi|^2$$

$$= \mu_{\mathrm{eff}}\, H \cdot H$$

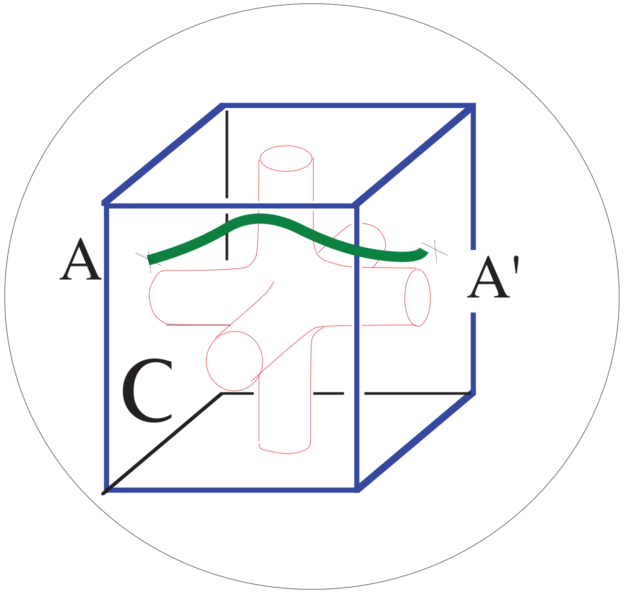
$$\mathrm{div}[\mu(H + \mathrm{grad}\,\varphi)] = 0$$

A better, symmetric, formulation

$\mathrm{rot}\, \mathbf{h} = 0$ $\qquad$ $\mathbf{b} = \mu \mathbf{h}$ $\qquad$ $\mathrm{div}\, \mathbf{b} = 0$

Both b and h C-periodic

$\langle \mathbf{h} \rangle = \mathbf{H}$ $\qquad$ $\langle \mathbf{b} \rangle = \mathbf{B}$



Find linear relation

$$\boxed{\mathbf{B} = \mu_{\mathrm{eff}}\, \mathbf{H}}$$

that allows non-trivial solution to exist

As a rule, μ_{eff} is a *matrix* (3×3)

Some theoretical justification is needed, some convergence result,

like

"when $\alpha \to 0$, exact solution

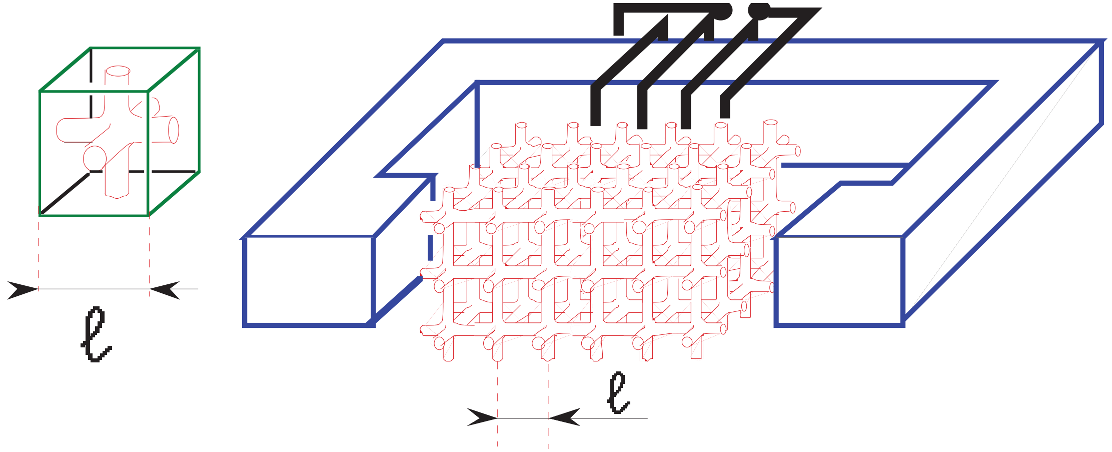
$h_\alpha = H + \text{grad}\ \varphi_\alpha$

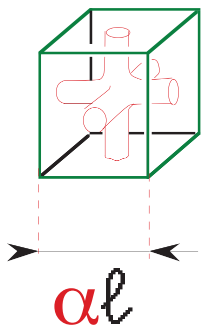
weakly converges to solution

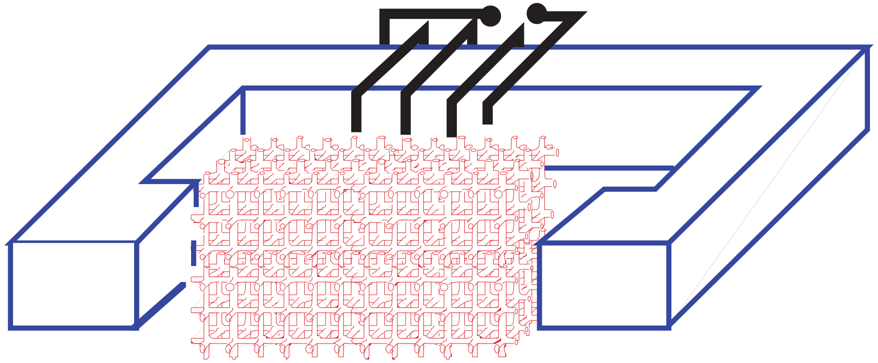
h of homogenized problem"

(the one in which μ_{eff} replaces μ_α), where "weakly" means that

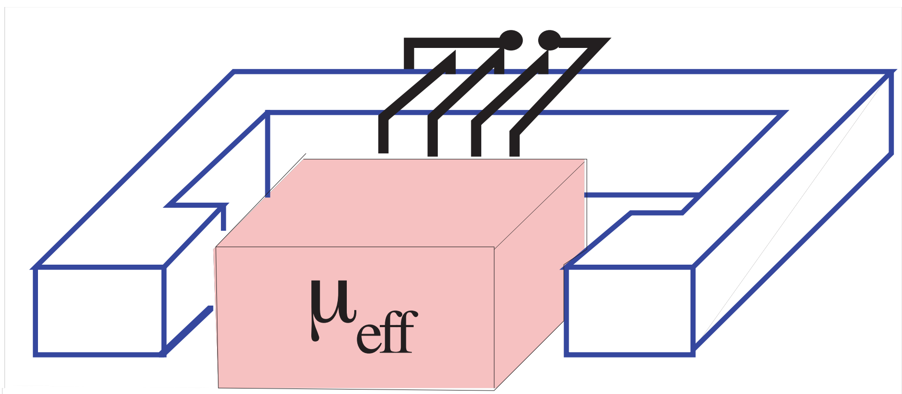
averages $\int h_\alpha \cdot b'$ and $\int \mu_\alpha h_\alpha \cdot h'$ converge for all test-fields h', b'



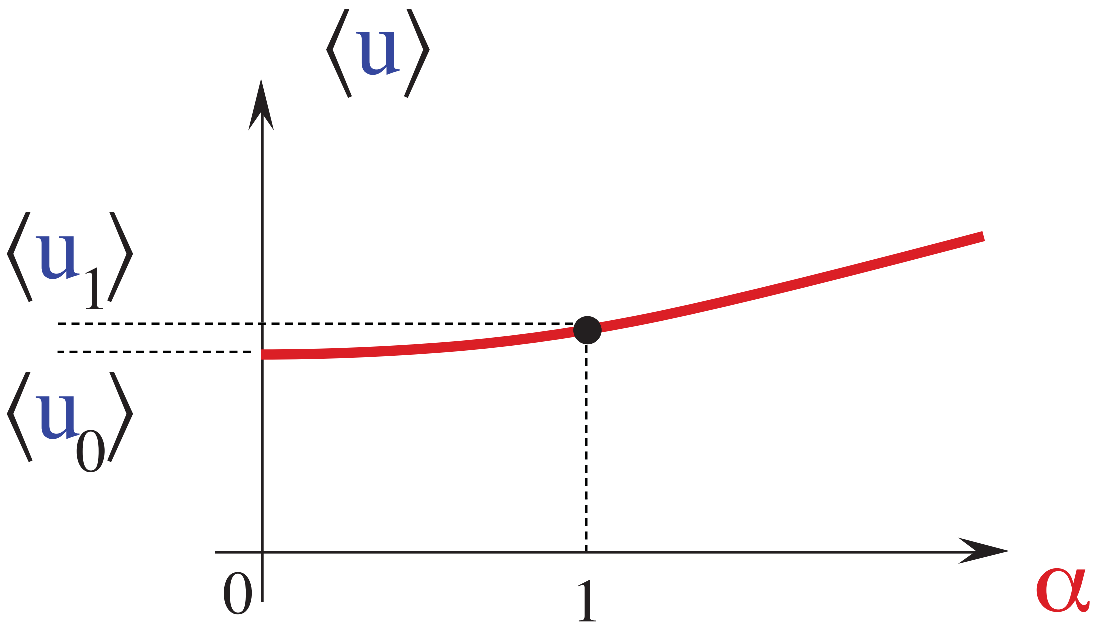




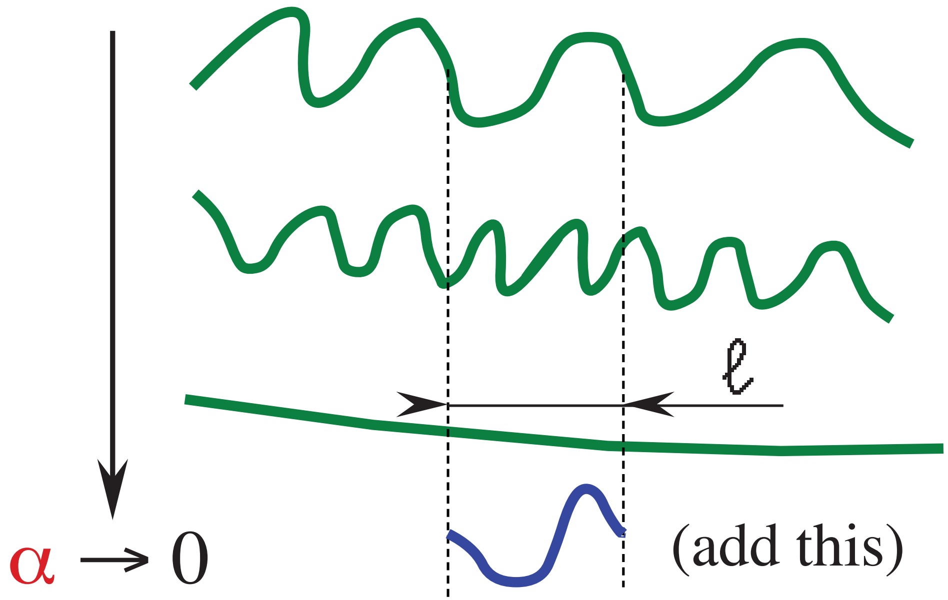
$$\alpha = \frac{\text{virtual cell-size}}{\text{actual cell-size}}$$



So, embed actual problem ("problem P") in family of virtual problems ("P_α", with P one of them, for instance P_1), and prove solution u_α weakly convergent to solution u_0 of some problem P_0, *simpler than* P_1.*
Then solve P_0.



Local corrections may be needed:



* Homogenization thus belongs to the larger family of *perturbative* techniques

$\boxed{\alpha \text{ often called "small parameter"}}$

Essential: α dimensionless quantity

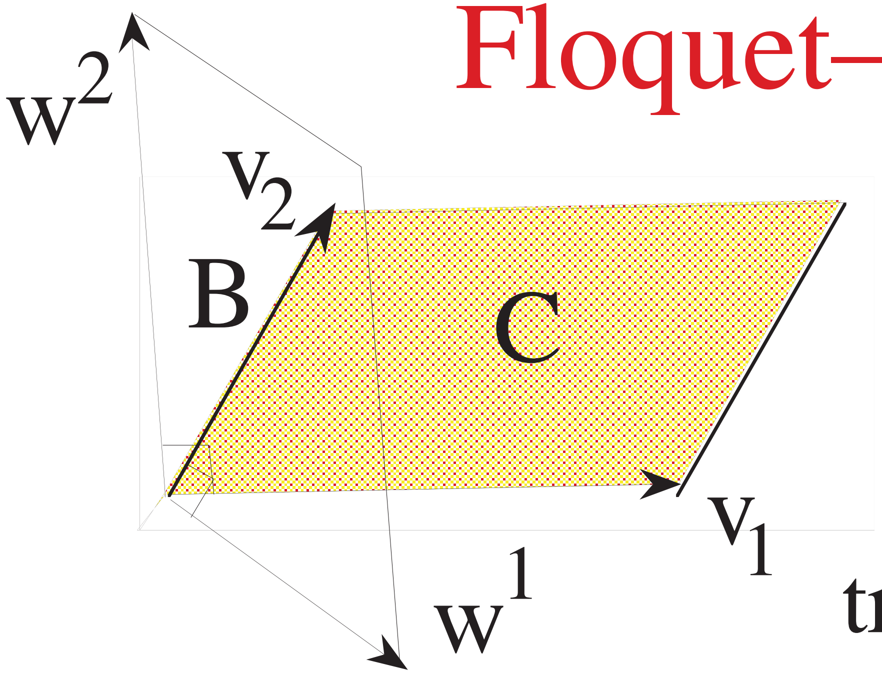
Non-essential: $\alpha = 1$ for actual problem.

Depends on reference value used: here, size of actual cell, but could be any problem-specific length, such as wavelength λ, or size L of the macroscopic device.

Since $\langle u_\alpha \rangle \sim \langle u_0 \rangle + \alpha\, \partial_\alpha \langle u_\alpha \rangle \big|_{\alpha = 0}$, what need be small is $\alpha \dfrac{\partial_\alpha \langle u_\alpha \rangle}{\langle u_\alpha \rangle}\Big|_{\alpha = 0}$, easy to estimate.

Most often (but not always!), can indeed be proven small (i.e., << 1) when ℓ/λ or ℓ/L << 1, the usual requisites.

Floquet–Bloch decomposition:



$$\boxed{v_i \cdot w^j = 2\pi\, \delta_i^j}$$

$$\boxed{\mathrm{vol}(B)\ \mathrm{vol}(C) = (2\pi)^3}$$

translation vectors $\tau = \sum_{i=1}^{i=3} z^i v_i,$

integer z_i

$$\hat{\varphi}_\kappa(x) = \mathrm{vol}(C) \sum_\tau e^{-i\kappa\cdot(\tau + x)} \varphi(x + \tau)$$

Notice: "Bloch components" $\hat{\varphi}_\kappa$ are C-periodic

$$\varphi(x) = \frac{1}{(2\pi)^3} \int_B d\kappa\ e^{i\kappa\cdot x}\, \hat{\varphi}_\kappa(x)$$

$\kappa \in B$ (akin to "Brillouin zone")

$\exp(i\kappa\cdot\tau) = \exp(i(\kappa + \sum_j \zeta_j w^j) \cdot \tau)$ for integer ζ_j, B a 3-*torus*.

Bloch and Fourier

$$\hat{\varphi}(\kappa) = \int d\tau\, e^{-i\kappa\cdot\tau}\, \varphi(\tau)$$

$$\varphi(x) = (2\pi)^{-3} \int d\kappa\, e^{i\kappa\cdot x}\, \hat{\varphi}(\kappa)$$

Fourier

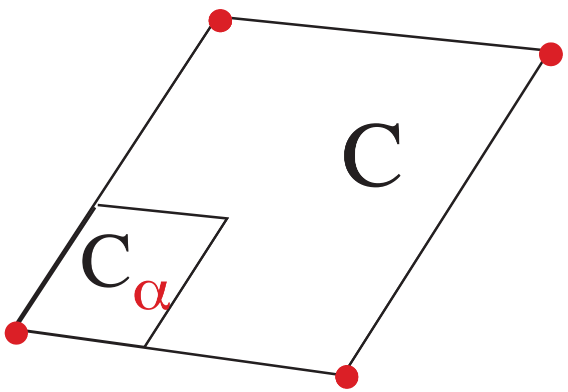
$$\hat{\varphi}_\kappa(x) = \mathrm{vol}(C) \sum_\tau e^{-i\kappa\cdot(\tau+x)}\, \varphi(x+\tau)$$

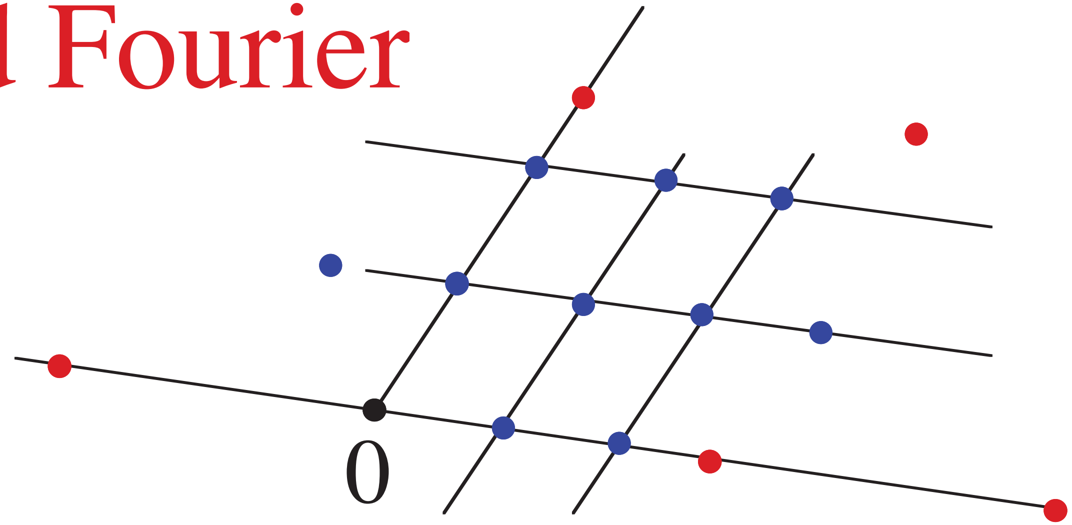
$$\varphi(x) = \frac{1}{(2\pi)^3} \int_B d\kappa\, e^{i\kappa\cdot x}\, \hat{\varphi}_\kappa(x)$$

Bloch

Intuitively clear, but not so obvious to formalize, connections between them

Bloch and Fourier





$C_\alpha = \alpha C$ (homothety)

T: • translation vectors

T_α : • translation vectors

$$\hat{\varphi}^{\alpha}_{\kappa}(x) = \mathrm{vol}(C_\alpha) \sum_{\tau \in T_\alpha} e^{-i\kappa\cdot(x+\tau)} \varphi(x+\tau) \xrightarrow[\alpha \to 0]{} \underbrace{\int dy\, e^{-i\kappa\cdot y}\, \varphi(y)}_{\hat{\varphi}(\kappa)}$$

(for regular φ)

$\boxed{\text{So the average } \langle \hat{\varphi}^{\alpha}_{\kappa} \rangle_{C_\alpha} \text{ over } C_\alpha \text{ tends to } \hat{\varphi}(\kappa) \text{ when } \alpha \to 0}$

There is a kind of reciprocal property:

- Bounded family φ^{α} of functions $\mathbb{L}^2(R^n)$
- Bloch representation $\{\hat{\varphi}_{\kappa}^{\alpha}, \kappa \in B_{\alpha}\}$ of each φ^{α}
- Function φ in $\mathbb{L}^2(R^n)$, its Fourier transform $\hat{\varphi}$

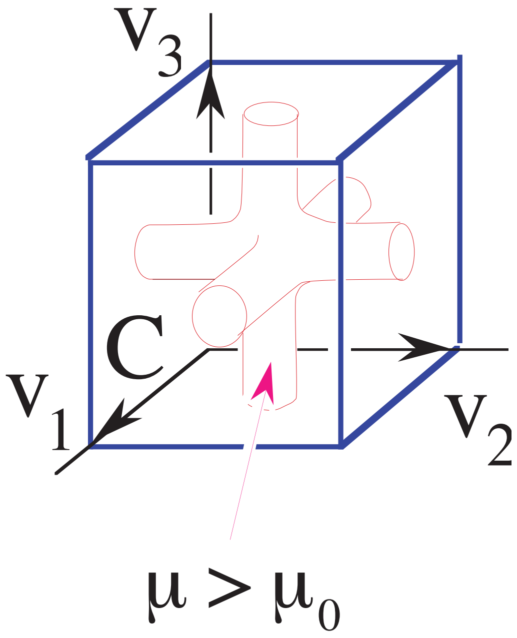
Theorem:

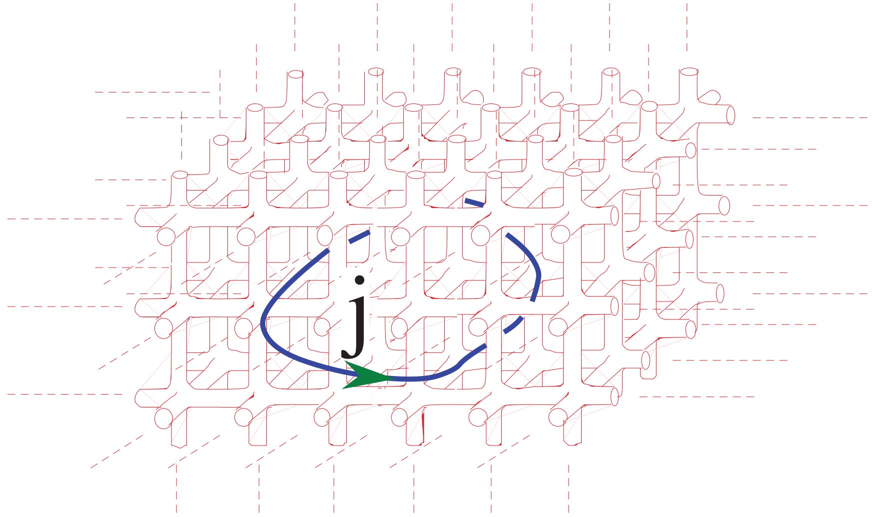
If $\boxed{\lim_{\alpha \to 0} \langle \hat{\varphi}_{\kappa}^{\alpha}(x) \rangle_{C_{\alpha}} = \hat{\varphi}(\kappa)}$ for all κ

Then $\boxed{\varphi^{\alpha} \rightharpoonup \varphi}$ when $\alpha \to 0$

(weak convergence, i.e.,

$$\int dx\, \varphi^{\alpha}(x)u(x) \xrightarrow{\alpha \to 0} \int dx\, \varphi(x)u(x) \quad \forall u)$$





Lattice T of translations τ leaving μ unchanged

Bloch decomposition:

$$h(x) = \frac{1}{(2\pi)^3} \int_B d\kappa \, e^{i\kappa \cdot x} \, \hat{h}_\kappa(x), \text{ etc.}$$

$$\begin{aligned} \text{rot } h &= j \\ b &= \mu \, h \\ \text{div } b &= 0 \end{aligned} \quad \Longrightarrow \quad \text{for each } \kappa, \quad \begin{aligned} (\text{rot} + i\,\kappa \times)\, \hat{h}_\kappa &= \hat{j}_\kappa \\ \hat{b}_\kappa &= \mu \, \hat{h}_\kappa \\ (\text{div} + i\,\kappa \cdot)\, \hat{b}_\kappa &= 0 \end{aligned}$$

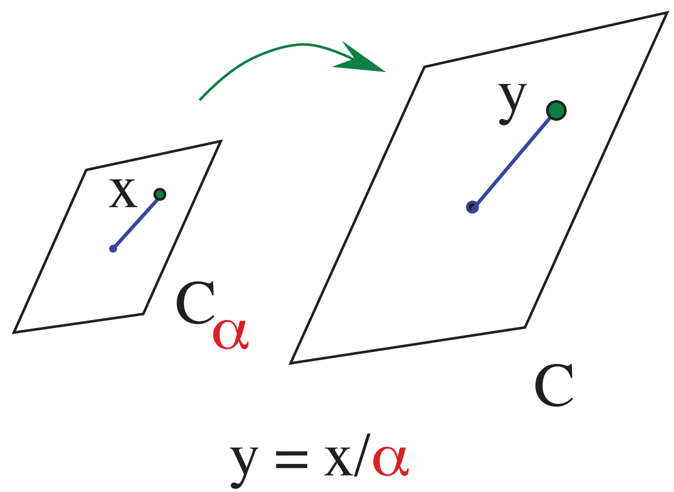
Embed problem in family, indexed on α, with cell C_α

$$\hat{h}^\alpha_\kappa(x) = \mathrm{vol}(C_\alpha) \sum_{\tau \in T_\alpha} e^{-i\kappa\,(x+\tau)}\, h^\alpha(x+\tau), \text{ etc.}$$

$$(\mathrm{rot} + i\,\kappa\,\times)\,\hat{h}^\alpha_\kappa = \hat{j}_\kappa$$
$$\hat{b}^\alpha_\kappa = \mu\,\hat{h}^\alpha_\kappa$$
$$(\mathrm{div} + i\,\kappa\,\cdot)\,\hat{b}^\alpha_\kappa = 0$$

All functions C_α-periodic.

$$h^\alpha(x) = \frac{1}{(2\pi)^3}\int_{B_\alpha} d\kappa\; e^{i\kappa\cdot x}\;\hat{h}^\alpha_\kappa(x)$$

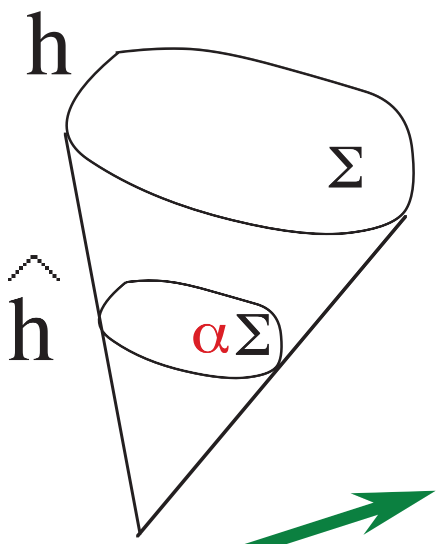


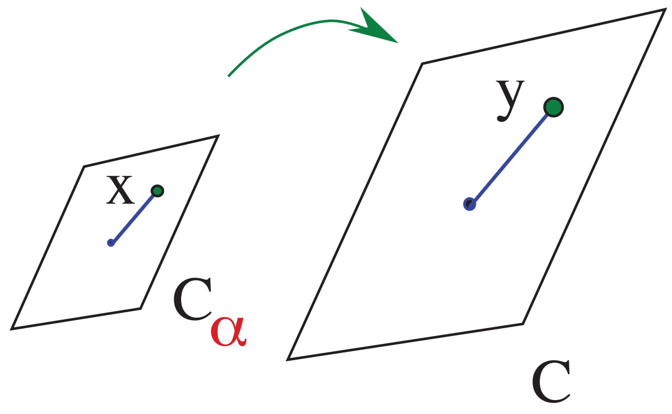
To study the $\alpha = 0$ limit, one must be able to compare the $\hat{h}^\alpha_\kappa$'s for *different* α's. "Pull back" to common domain C, by scaling, to let them all live on the *same* reference cell.

Scaling: "Pull back" Bloch components $\hat{h}_\kappa$, etc., to C, call the pullbacks h_κ, b_κ, etc.

$$h^\alpha_\kappa(y) \overset{\text{def.}}{=} \hat{h}^\alpha_\kappa(\alpha y)$$

$$\hat{h}^\alpha_\kappa(x) = h^\alpha_\kappa(x/\alpha)$$





$y = x/\alpha$

Observe that

$$\operatorname{rot} \hat{h}^\alpha_\kappa(x) = \frac{1}{\alpha} \operatorname{rot} h^\alpha_\kappa(y)$$

(chain-rule, or Stokes thm.)

Same with div

$$(\operatorname{rot} + i\alpha\kappa \times) h^\alpha_\kappa = \alpha j^\alpha_\kappa$$

$$b^\alpha_\kappa = \mu\, h^\alpha_\kappa$$

$$(\operatorname{div} + i\alpha\kappa \cdot) b^\alpha_\kappa = 0$$

New cell-problems, now all set on C

Note that $\langle \hat{h}^\alpha_\kappa \rangle_{C_\alpha} = \langle h^\alpha_\kappa \rangle_C$

$$(\mathrm{rot} + i\alpha\kappa\,\times)h^\alpha_\kappa = \alpha j^\alpha_\kappa \quad\Longrightarrow\quad i\kappa \times \langle h^\alpha_\kappa \rangle_C = \langle j^\alpha_\kappa \rangle_C \to \hat{j}(\kappa)$$

$$b^\alpha_\kappa = \mu\, h^\alpha_\kappa$$

$$(\mathrm{div} + i\alpha\kappa\,\cdot)b^\alpha_\kappa = 0 \quad\Longrightarrow\quad i\kappa \cdot \langle b^\alpha_\kappa \rangle_C = 0$$

Limits H and B of $\langle h^\alpha_\kappa \rangle$ and $\langle h^\alpha_\kappa \rangle$ satisfy

$$\begin{cases} i\kappa \times H = \hat{j}(\kappa) \\ B = \mu_{eff} H \\ i\kappa \cdot B = 0 \end{cases}$$

$\alpha \to 0$: $\Downarrow$

the same system as Fourier coeffts for $\{h, b\}$ in the homogeneous case:

$$\mathrm{rot}\, h = 0 \qquad \langle h \rangle = H$$

$$b = \mu\, h \qquad B = \mu_{eff} H$$

$$\mathrm{div}\, b = 0 \qquad \langle b \rangle = B$$

b and h C-per.

$$\boxed{\begin{aligned} & i\kappa \times \hat{h}(\kappa) = \hat{j}(\kappa) \\ & \hat{b}(\kappa) = \mu_{eff}\, \hat{h}(\kappa) \\ & i\kappa \cdot \hat{b}(\kappa) = 0 \end{aligned}}$$

so $\{h^\alpha, b^\alpha\} \rightharpoonup \{h, b\}$

So indeed,

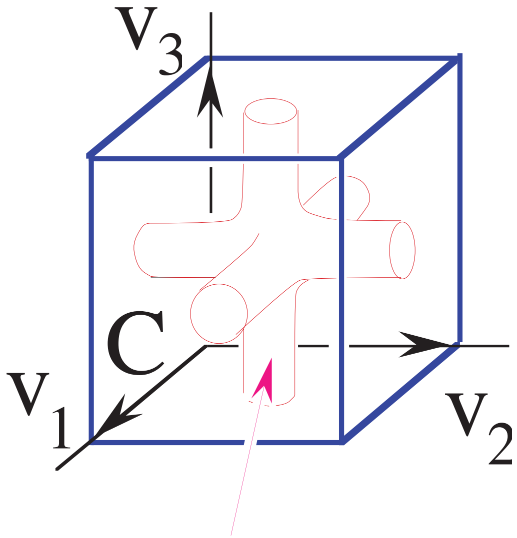
The weak $\alpha = 0$ limit inherent in Bloch

provides the expected convergence result.

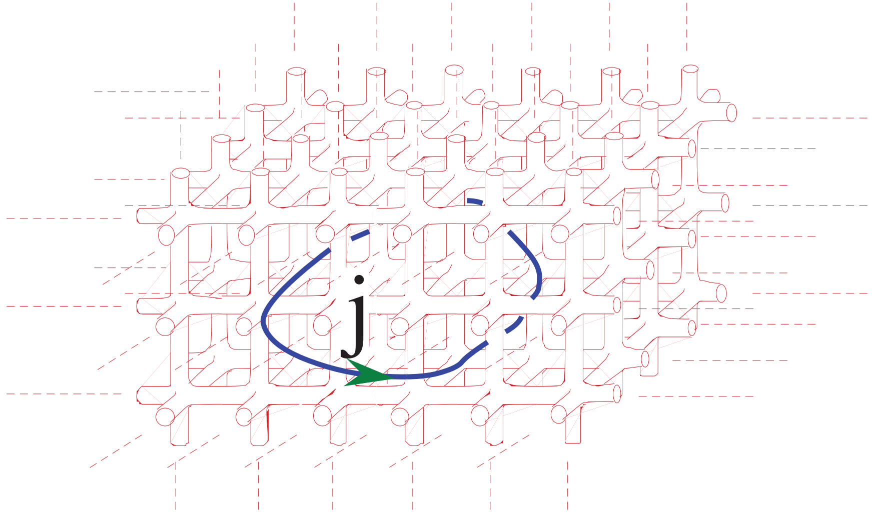
Practical benefit:

Different sub-problems (one for each κ)
reduce to a single "cell problem",
that yields effective μ.

Now, AC source



$\mu \neq \mu, \ \varepsilon = \varepsilon_0 - i\sigma/\omega$



$$-i\omega d + \mathrm{rot}\, h = j$$

$$d = \varepsilon\, e, \ b = \mu\, h$$

$$i\omega b + \mathrm{rot}\, e = 0$$

ε, μ unchanged by translations $\tau \in T$

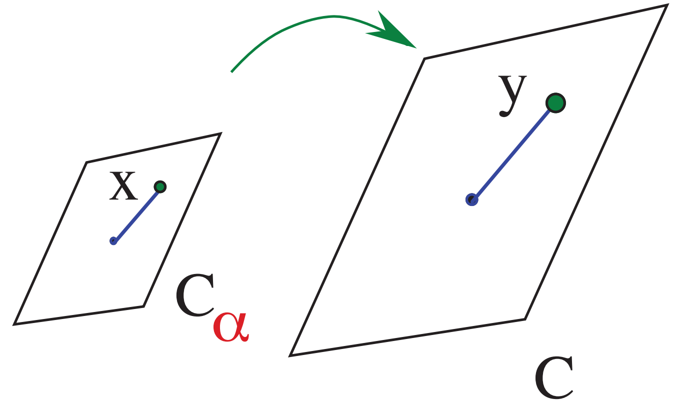
$$-i\omega \hat{d}_\kappa + (\mathrm{rot} + i\kappa\times)\, \hat{h}_\kappa = \hat{j}_\kappa$$

$$\hat{d}_\kappa = \varepsilon\, \hat{e}_\kappa, \ \hat{b}_\kappa = \mu\, \hat{h}_\kappa$$

$$i\omega \hat{b}_\kappa + (\mathrm{rot} + i\kappa\times)\, \hat{e}_\kappa = 0$$

for each $\kappa \in B$

Scaling: $h^{\alpha}_{\kappa}(y) \overset{\text{def.}}{=} \hat{h}^{\alpha}_{\kappa}(\alpha y)$, etc.



$y = x/\alpha$

$$-i\omega\alpha d^{\alpha}_{\kappa} + (\mathrm{rot} + i\alpha\kappa\,\times) h^{\alpha}_{\kappa} = \alpha j^{\alpha}_{\kappa}$$

$$d^{\alpha}_{\kappa} = \varepsilon\, e^{\alpha}_{\kappa}, \quad b^{\alpha}_{\kappa} = \mu\, h^{\alpha}_{\kappa}$$

$$i\omega\alpha b^{\alpha}_{\kappa} + (\mathrm{rot} + i\alpha\kappa\,\times) e^{\alpha}_{\kappa} = 0$$

$$(\mathrm{div} + i\alpha\kappa\,\cdot) d^{\alpha}_{\kappa} = \alpha q^{\alpha}_{\kappa}$$

$$(\mathrm{div} + i\alpha\kappa\,\cdot) b^{\alpha}_{\kappa} = 0$$

($i\omega q + \mathrm{div}\, j = 0$ results in $i\omega\alpha q^{\alpha}_{\kappa} + \mathrm{div}\, j^{\alpha}_{\kappa} = 0$)

$$-i\omega\alpha d + (\mathrm{rot} + i\alpha\kappa\times)h_\kappa^\alpha = \alpha j_\kappa^\alpha \Rightarrow$$

$$-i\omega\langle d_\kappa^\alpha\rangle_C + i\kappa\times\langle h_\kappa^\alpha\rangle_C = \langle j_\kappa^\alpha\rangle_C$$

$$b_\kappa^\alpha = \mu\, h_\kappa^\alpha$$

$$\quad D \qquad\qquad H \qquad\qquad \downarrow$$

$$(\mathrm{div} + i\alpha\kappa\cdot)b_\kappa^\alpha = 0 \quad\Rightarrow\quad i\kappa\cdot\langle b_\kappa^\alpha\rangle_C = 0 \qquad\qquad \hat{j}(\kappa)$$

$$B$$

$$-i\omega D + i\kappa\times H = \hat{j}(\kappa)$$

$$B = \mu_{\mathrm{eff}} H$$

$$\alpha \to 0:$$

$$i\omega B + i\kappa\times E = 0$$

$$D = \varepsilon_{\mathrm{eff}} E$$

$$\mathrm{rot}\, h = 0 \qquad \langle h\rangle = H$$

$$b = \mu\, h \qquad B = \mu_{\mathrm{eff}} H$$

$$\mathrm{div}\, b = 0 \qquad \langle b\rangle = B$$

b and h C-per.

same for e and d

$$-i\omega\hat{d}(\kappa) + i\kappa\times\hat{h}(\kappa) = \hat{j}(\kappa)$$

$$\hat{d}(\kappa) = \varepsilon_{\mathrm{eff}}\,\hat{e}(\kappa)$$

$$i\omega\,\hat{b}(\kappa) + i\kappa\times\hat{e}(\kappa) = 0$$

$$\hat{b}(\kappa) = \mu_{\mathrm{eff}}\,\hat{h}(\kappa)$$

So, it seems,

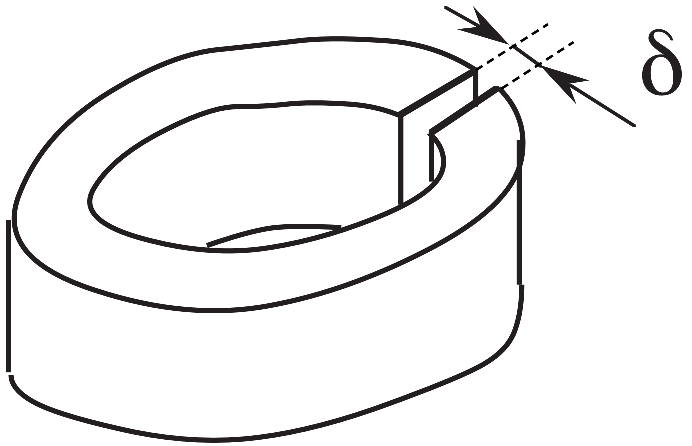
the method does apply to full Maxwell,

but with disappointing results:

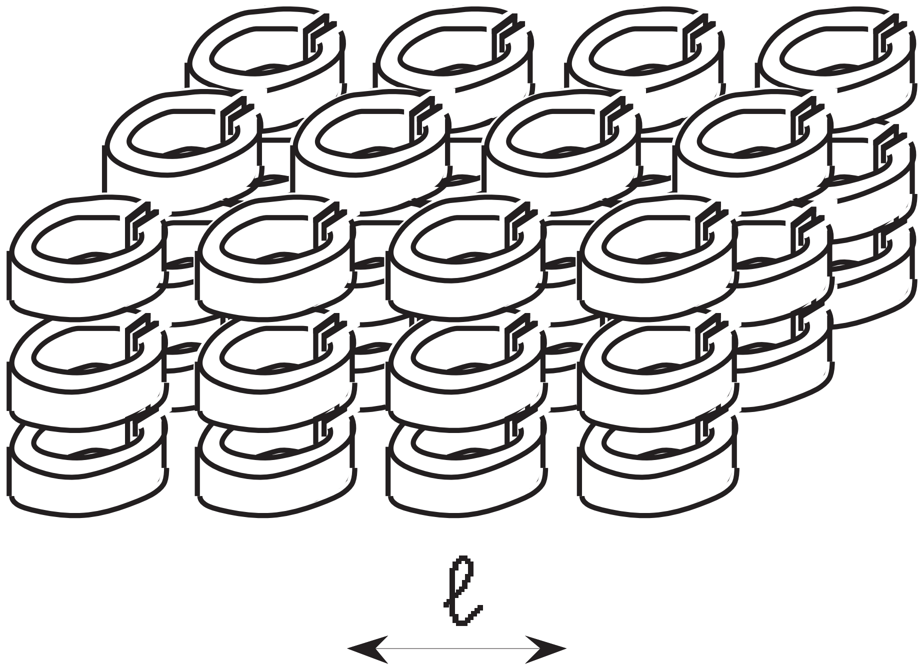
Effective ε and μ are the static ones

(no chirality, no negative index).

The introduction
of a second small parameter, "competing" with α,
will save the day.



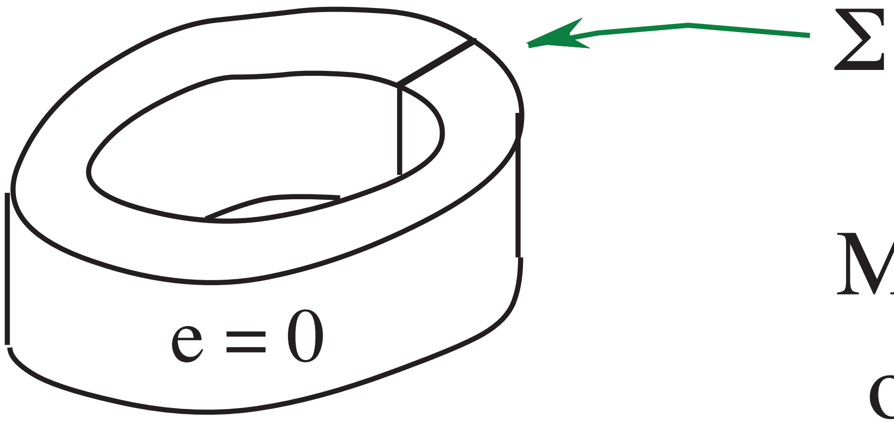
(very good conductor)



First small parameter: $\ell << \lambda$

Second small parameter: $\delta << \ell$

(to say nothing of "contrast", $\omega\varepsilon_0/\sigma$, here taken = 0)



(perfect conductor)

Model by "capacitive layer" on slit Σ and its translates

Weak formulation in h:

Σ = set-union of all slits

A = "air" region (outside rings)

$$\int_A i\omega\mu\, h \cdot h' + \int_A \frac{1}{i\omega\varepsilon}(\mathrm{rot}\, h - j) \cdot \mathrm{rot}\, h' + \ldots$$

$$+\int_\Sigma \frac{\delta}{i\omega\varepsilon}(n \cdot \mathrm{rot}\, h)\,(n \cdot \mathrm{rot}\, h') = 0 \quad \forall\, h' \ \text{("test field")}$$

Weak formulation in h:

Σ = set-union of all slits

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$$\int_A i\omega\mu\, h \cdot h' + \int_A \frac{1}{i\omega\varepsilon}(\mathrm{rot}\, h - j) \cdot \mathrm{rot}\, h' + \ldots$$

$$+ \int_\Sigma \frac{\delta}{i\omega\varepsilon}(n \cdot \mathrm{rot}\, h)\,(n \cdot \mathrm{rot}\, h') = 0 \quad \forall\, h' \ (\text{"test field"})$$

Embedding:

$$\int_{A_\alpha} i\omega\mu\, h^\alpha \cdot h' + \int_{A_\alpha} \frac{1}{i\omega\varepsilon}(\mathrm{rot}\, h^\alpha - j) \cdot \mathrm{rot}\, h' + \ldots$$

$$+ \int_{\Sigma_\alpha} \frac{\alpha^3 \delta}{i\omega\varepsilon}(n \cdot \mathrm{rot}\, h^\alpha)\,(n \cdot \mathrm{rot}\, h') = 0 \quad \forall\, h'$$

Scaling:

(Now A and S refer to the reference cell, h^{α} is the

κ-Bloch component, C-periodic)

$$\alpha^3 \int_A i\omega\mu\, h^{\alpha} \cdot h'$$

$$+ \alpha \int_A \frac{1}{i\omega\varepsilon} (\mathrm{rot} + i\alpha\kappa \times)\, h^{\alpha} - \alpha j) \cdot (\mathrm{rot} - i\alpha\kappa \times)\, h'$$

$$+ \alpha^3 \int_{\Sigma} \frac{\delta}{i\omega\varepsilon} (n \cdot (\mathrm{rot} + i\alpha\kappa \times)\, h\,)\,(n \cdot (\mathrm{rot} - i\alpha\kappa \times)\, h')$$

$$= 0 \quad \forall\, h'$$

just right to preserve resonance

The $\alpha = 0$ limit: An exotic cell problem

Find $\varphi \in \Phi^H$ such that, $\forall\ \varphi' \in \Phi^0$,

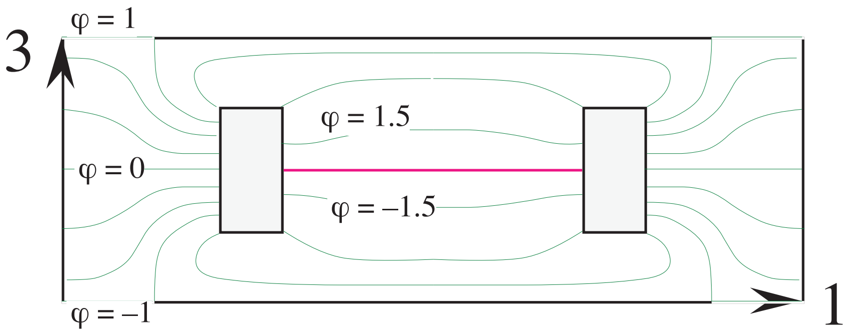
$$\int_A i\omega\mu\ \nabla\varphi \cdot \nabla\varphi' + \frac{1}{i\omega C}[\varphi][\varphi'] = 0$$

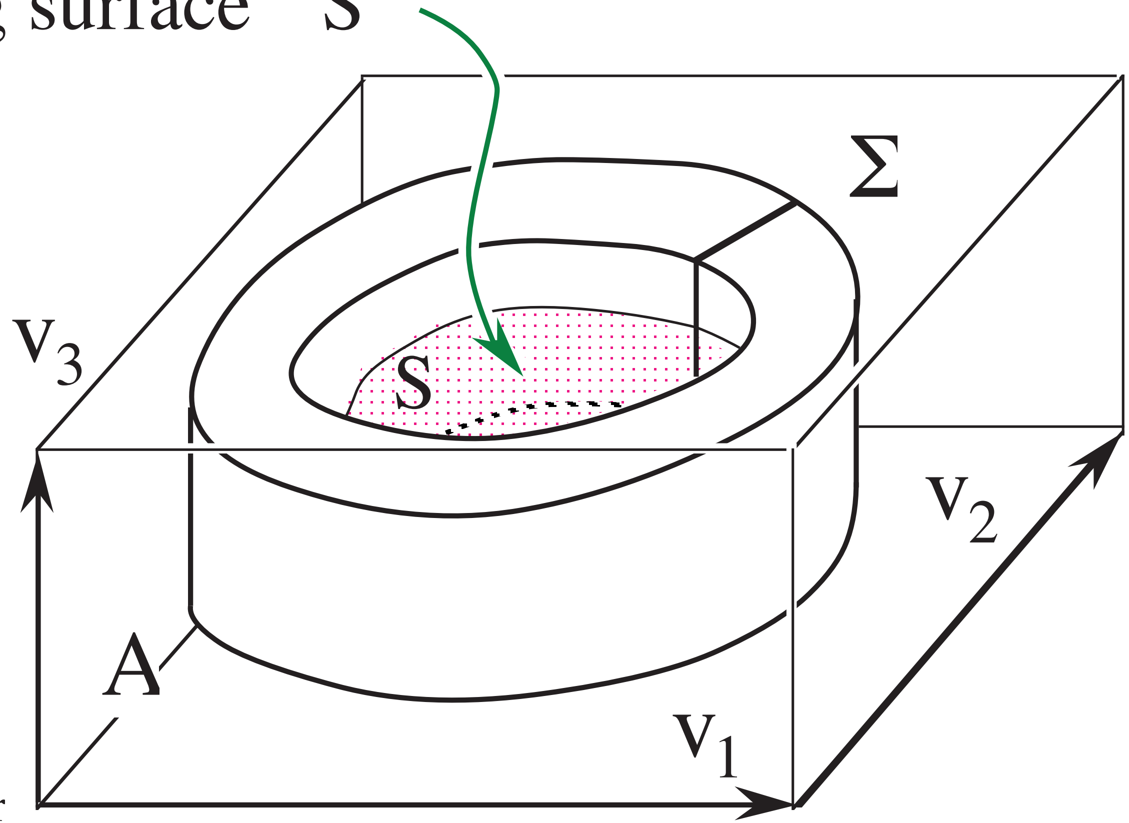
where $C = \int_\Sigma \frac{\varepsilon}{\delta}$

(capa of slit)

and $\Phi = \{\varphi :\ \ \varphi(x + t) - \varphi(x) = H \cdot \tau \ \text{ for } \ \tau = v_1, v_2, v_3\}$

with "jump" $[\varphi]$ across "cutting surface" S





Then, $\mu_{eff}\, H \cdot H =$

$$\int_A \mu\, |\nabla\varphi|^2 - \frac{1}{\omega^2 C}[\varphi]^2$$

Will get complex part if finiteness of σ is accounted for

Thanks